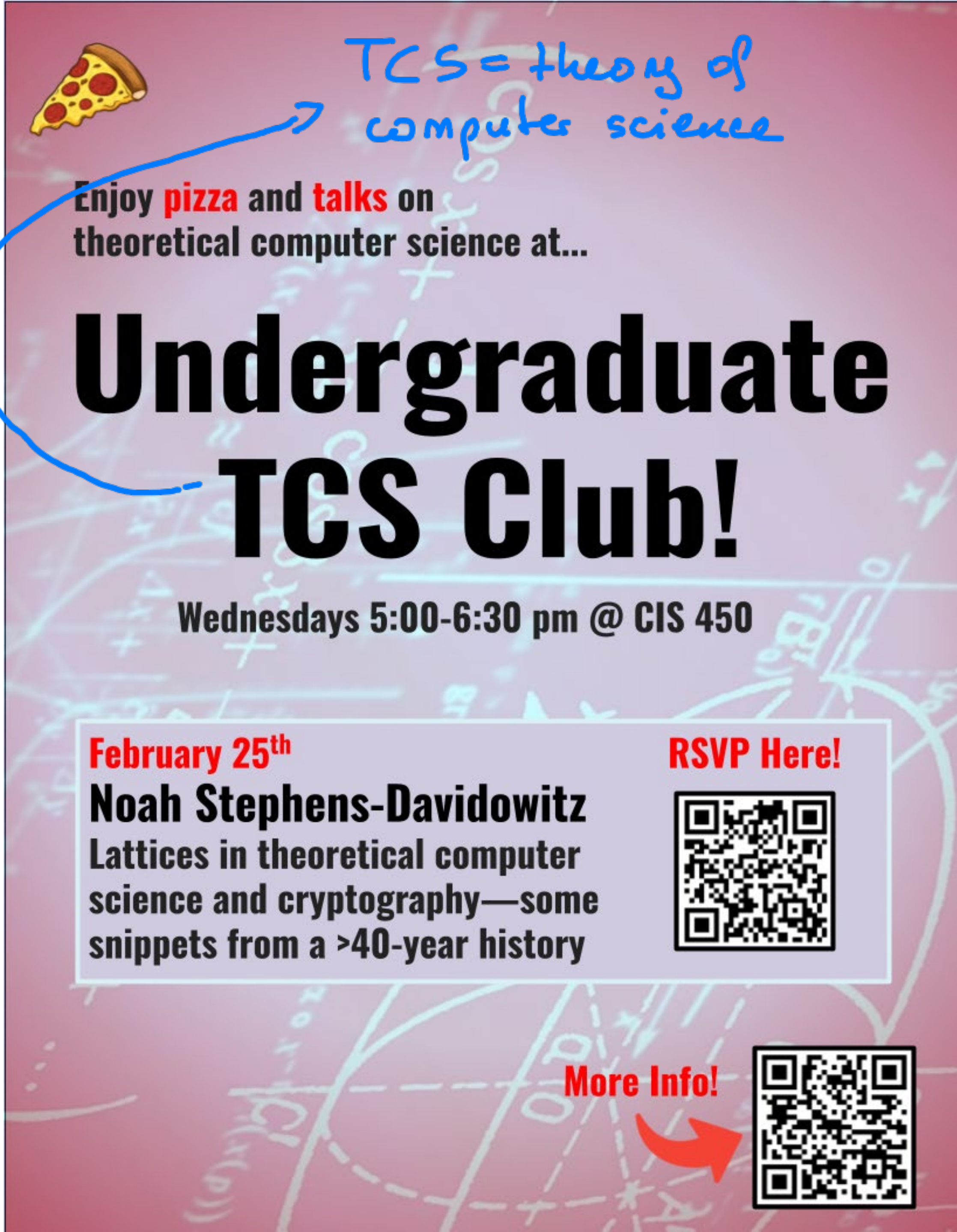
Announcements

- Regrade requests are due a week after we open the grades. this Wednesday for the prelim
- Updated course policies on the Web to include all policies (dropped quiz, regrade deadline, etc)
- Monday & Tuesday section: practice with flows and stable matching, no quiz

10am office hour today is on zoom

TCS = theory of computer science

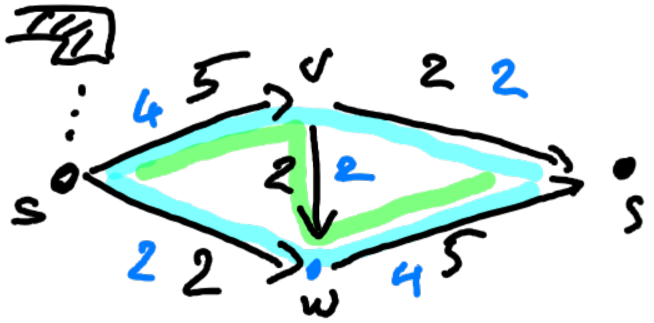


The maximum flow problem

Input: directed graph $G=(V,E)$ $\quad s,t \in V$ source & sink

capacity $c_e \geq 0$ all $e \in E$

assume s has no entering edges



flow $\quad 0 \leq f_e \leq c_e \quad$ capacity constraint

$\forall v \neq s,t \quad \sum_{e \text{ enters } v} f_e = \sum_{e \text{ leaves } v} f_e \quad$ flow conservation

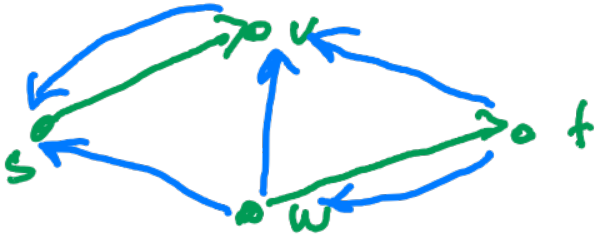
← flow entering = flow leaving

blue = flow

goal: maximize flow value

$\text{value}(f) = \sum_{v:\, sv \in E} f_{sv}$ = flow leaving s

Residual graph



forward edge

backwards

Ford Fulkerson algorithm

Set $f_e = 0 \ \forall e$

While there is s-t path in E_f

find path P

$$\delta = \min\left(\min_{\substack{e \in P \\ \text{forward}}} (c_e - f_e), \min_{\substack{e \in P \\ \text{backwards}}} f_e\right)$$

$$f_e = \begin{cases} f_e + \delta & \text{if } e \in P \text{ forward} \\ f_e - \delta & \text{if } e \in P \text{ backwards} \\ f_e & \text{otherwise} \end{cases}$$

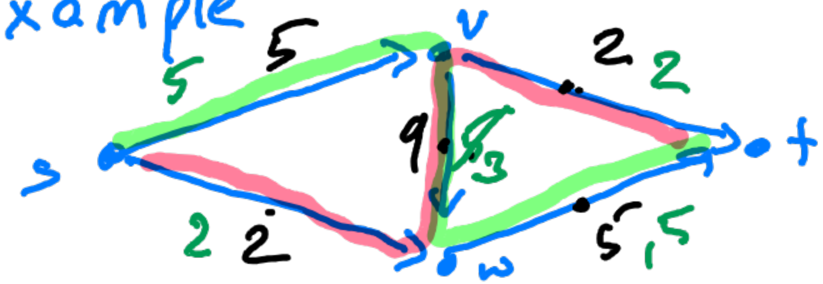
endwhile

f residual graph (V, E_f)

$$E_f = \{ (v,w) \text{ s.t. } (v,w) \in E \ \& \ f_{vw} < c_{vw}$$
$$\text{or} \quad (w,v) \in E \ \& \ f_{wv} > 0 \}$$

Example

c_e



Run of alg.

1. path s v w t $\quad \delta = 5$

2. E_f

path s w v t

$\delta = 2$

3. E_f

no path.

Properties Ford Fulkerson algorithm and run time

Seen last class:

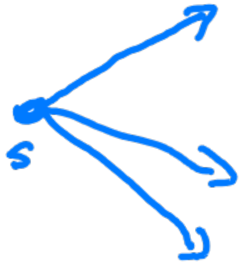
1. Maintains valid flow ✓
2. Each iteration increases flow value with δ ✓
3. If all capacities integer then flow is always integer valued ✓

Running time

work per iteration $m = |E|$ & $n = |V|$

$$\left.\begin{array}{ll} O(m) & \text{create } E_f \\ O(m+n) & \text{find path} \end{array}\right\} O(m+n)$$

$O(n)$ augment = increase flow



iterations: if c_e integer all e f $C = \sum_{sv \in E} c_{sv}$

$\Rightarrow$ δ be integer $\Rightarrow$ max # iterations $C+1$

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Example of flows. Are they of maximal value?

A. Neither is

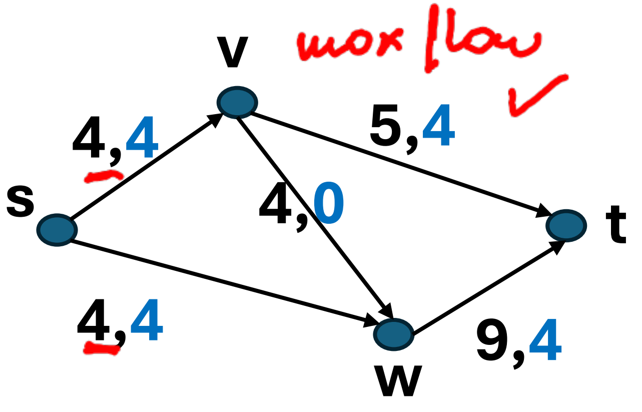
B. Flow A is but flows B and C are not

C. Flows A and B are but flow C is not

D. All are of maximum value

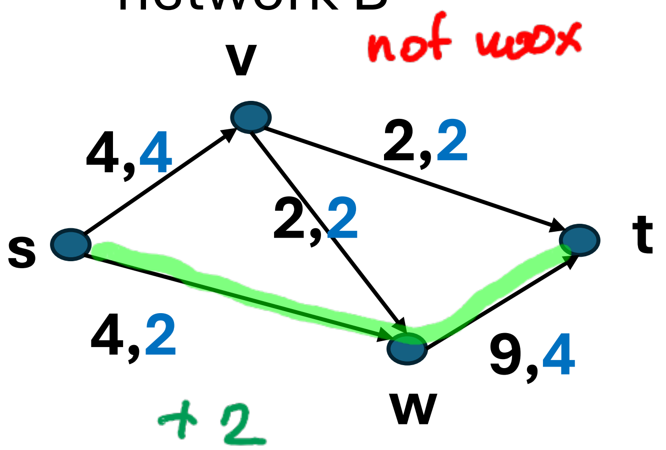
E. none of these true

network A



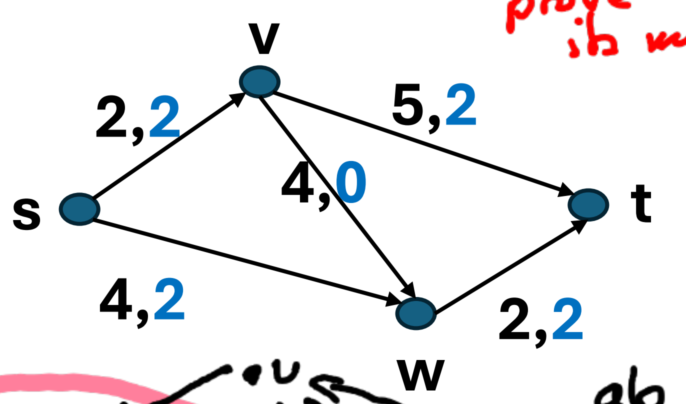
capacity flow

network B



network C

hope to prove its max

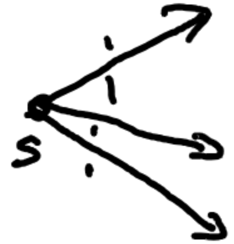


alg stuck

Cuts and where to measure the flow value

Alg ends   no s–t path

$A = \{ v : \exists\ s \to v \text{ path in } E_f \}$ $s \in A$, $t \notin A$



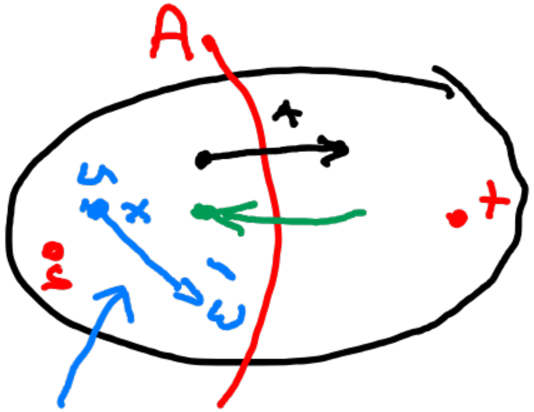
value$(f) = \sum_{sv \in E} f_{sv}$

$$\text{value}(f) = \sum_{sv \in E} f_{sv} = \sum_{sv \in E} f_{sv} + \sum_{v \in A \setminus s} \Big(\sum_{vw \in E} f_{vw} - \sum_{wv \in E} f_{wv} \Big)$$

↑ 0 due to flow conservation
flow out − flow in

$$= \sum_{\substack{v \in A \\ w \notin A \\ (v,w) \in E}} f_{vw} - \sum_{\substack{v \in A \\ w \notin A \\ (w,v) \in E}} f_{wv}$$

− flow leaving A − flow entering A



included + & − so cancel

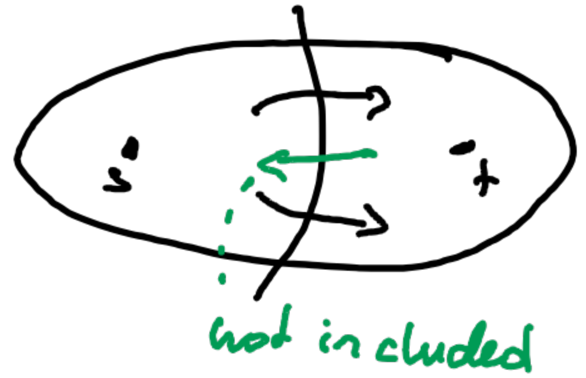
$$\leq \sum_{v \in A,\, w \notin A} c_{vw} - 0$$

~ capacity of cut

Flow value at the end of the Ford-Fulkerson alg

given set A, $B = V \setminus A$ $\quad s \in A$, $t \notin A$

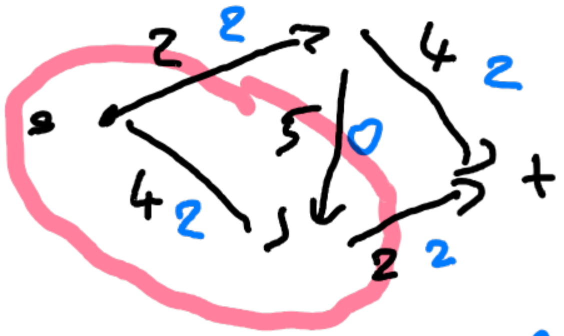
$$\text{capacity}(A,B) = \sum_{\substack{v \in A \\ w \in B}} c_{vw}$$



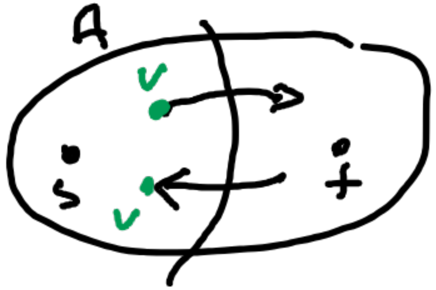
We proved above

value$(f) \le$ capacity(A,B) true all flow & all cuts

C network



A = reachable from s in last residual graph



$(v,w) \in E$, $v \in A$, $w \notin A \Rightarrow f_{vw} = c_{vw}$

$(w,v) \in E$ $v \in A$, $w \notin A \Rightarrow f_{wv} = 0$

edge not residual graph as w is not reachable

At end we have equal in equations above

Maximum Flows and Minimum Cuts

value$(f) \leq$ capacity(A,B) all flow f & all (A,B) $s \in A, t \in B$

Ford-Fulkerson finds flow & cut
value$(f) =$ capacity(A,B)

$\Rightarrow$ f is max flow

$\Rightarrow$ (A,B) minimum capacity cut